© Kamla-Raj 2017 Int J Edu Sci, 17(1-3): 224-237 (2017)
DOI: 10.31901/24566322.2017/17.1-3.27

Using Questionnaires in the Learning of Congruency of Triangles to Incite Formal and Informal Reasoning

Deonarain Brijlall

Department of Mathematics, Durban University of Technology, South Africa
E-mail: deonarainb@dut.ac.za

KEYWORDS Transformation Geometry. Euclidean Geometry. Activity Design. Formal Reasoning. Informal Reasoning. Pedagogical Content Knowledge

ABSTRACT This qualitative study examined the influence learning material design has on the learning of geometry in the middle school curriculum. The middle school mathematics learners (n =82) involved were from two South African rural schools in the province of KwaZulu-Natal. The research was conceptualised in terms of Vygotsky's educational theory and the process of scaffolding. Researchers from a South African university designed questionnaires based on scaffolding guidelines suggested by Zhao and Orey. The questionnaires comprised of a series of geometry tasks which spanned two weeks. These questionnaires were administered to the learners and their written responses were analyzed. After analysis of these responses interviews were carried out to verify or refute the views of the researcher. Data yielded by these research instruments confirmed certain assumptions and literature claims. The study revealed that the intervention design effectively managed to integrate the two types of geometries to strengthen the concept of congruency of triangles. Many recommendations, for the teaching of congruency of triangles, emanated from the findings.

INTRODUCTION

Adler (2005) focused on the complex issues involved in the teaching and learning of mathematics. She felt it was important that we understood "how to make mathematics learnable by all children" (Adler 2005: 2). Her area of interest is to know more about the support and mathematical preparation that teachers receive and to make them more efficient and skilful in the classroom context. Even (1990) also subscribed to this notion and stated that the way mathematics is taught is important. The emphasis in recent times is to teach in such a manner that learners understand so that meaningful learning takes place. The researcher has been involved in many recent studies exploring the pedagogical knowledge of mathematics teachers (Bansilal et al. 2014; Brijlall 2011; Brijlall et al. 2012; Brijlall and Isaac 2011; Brijlall and Maharaj 2011, 2014, 2015). The role of the teacher is to assist the learner to understand the subject matter. With this in mind the researchers in this study embarked on the use of questionnaires to guide learning in geometry with the hope that conceptual understanding will occur. In order to provide guidance, with the least assistance from the teacher, the principle of scaffolding was employed in the design in the questionnaires. Similar strategies were employed in questionnaires by these researchers in aiding the learning of concepts in calculus. Such studies were carried out in South African higher education institutions (Brijlall 2011; Brijlall and Isaac 2011; Ndlovu and Brijlall 2016). However, this paper reports on a study which is different in that it is carried out in two rural schools in the province of KwaZulu-Natal in South Africa. It was found that these specially designed activities were appropriate in keeping with the current education system in South Africa which emphasizes that there should be a move towards learner-centred approaches being included in instructional strategies (DoE 2003).

Aim and Critical Question

The aim of this research was to generate thick interpretations of data in the design of questionnaires to promote scaffolding, written responses and interviews, in order to explore the influence of teaching design on learning of geometry. In questionnaire 1 the design involves *formal reasoning* which we define as problem solving involving the provision of evidence from axioms, lemmas, corollaries and theorems. If evidence is provided based on measurement and practical means then we regard the type of reasoning as *informal*. To unpack the research aim we asked: *How well do grade ten learners understand congruency of triangles when working with their transformations as compared to formal reasoning?* To address this fundamental research query, the following sub-questions were asked: (a) how well do grade ten learners understand the four cases of congruency? (b) to what extent do learners succeed in their conceptual understanding of congruency in Euclidean Geometry? (formal reasoning), (c) to what extent do learners succeed in their conceptual understanding of congruency in Transformation Geometry? (informal reasoning) and (d) how does informal reasoning make a difference in the conceptual understanding of congruency in geometry?

Research in Euclidean Geometry

Some South African research studies recommended changes in the teaching of geometry in order to improve understanding (Brijlall et al. 2006; De Villiers 2008; Mudaly 2007). Brijlall et al. (2006) looked at how learners' experiences in a Technology class could be used to inform the effective teaching of geometry. They recommended that Euclidean Geometry teachers build upon learners' intuitive or in-born knowledge of geometrical shapes (like triangles) in their teaching. This would facilitate the understanding of van Hiele's levels one and two: the naming of shapes and the analysis, respectively (van Hiele 1999).

It should be pointed out that Mathematics teachers need to master Euclidean Geometry knowledge before they can engage learners' learning through a "non-textbook" method. Adler (2005) argued that mathematics teachers need to have a conceptual understanding of the subject. With the intention of supporting conceptual understanding teachers can approach teaching through different Mathematical perspectives, depending on the context (Hall 1999).

In another South African study of the instructional strategies followed by grade eleven teachers, Mthembu (2007) found that many Mathematics teachers teach geometry using the non-constructivist model of the "teacher-talk" method. In countries like Belgium, teacher-centred approaches are still followed (Fagnant 2005). It is evident that, although many Mathematics teachers understand the constructivist approach to learning they tend not to implement it.

Research in Transformation Geometry

The researcher found few studies that looked at Transformation Geometry in the South African context. This shortfall may have occurred due to this section not having been a part of the Mathematics syllabus in the past. This section was introduced post 2005 and hence did not receive priority for research. International literature argues that congruent and similar triangles can be understood better in Transformation Geometry (Beevers 2001). Proofs of congruency in Transformation Geometry can be done by explanation instead of verification with younger children. Proving congruency by explanation is easier than the complications of Euclidean Geometry as any shape will be congruent to its image under translation, reflection and rotation (Fernandez 2005; Rival 1987). This proof by explanation is more convenient in the contemporary education system, where learners are encouraged to speak their minds about their observation of patterns.

History reveals that Transformation Geometry was once popular in the discovery of congruent patterns of geometrical shapes, for example tiling of pentagonal shapes (Rival 1987). Fernandez (2005) argued that Transformation Geometry (DoE 2003) is rich in possibilities of Mathematical investigation and discussions of "real-life" situations. The current South African mathematics syllabus includes Transformation Geometry. Some mathematics textbooks show designs of patterns generated through congruent figures. These patterns are used to design, *inter alia*, clothes and tiles, and are also incorporated into art exhibitions.

Proofs and Proving in High Schools

High school teachers are faced with the challenging task of teaching proofs. The researcher's personal experience as a Mathematics learner and teacher revealed that many learners did not understand or enjoy proofs of theorems. This problem became more pronounced in the proving of Euclidean Geometry riders. Mudaly (2007) compiled research findings showing that learners had been performing poorly in Euclidean

Geometry in the past two decades. The KwaZulu-Natal Department of Education (KZN DoE 2004) revealed similar findings in recent years. Sometimes this poor performance results in an unpleasant atmosphere in the Mathematics classroom. Indeed, as Davis and Hersh (1983) note, many teachers become frustrated when learners do not understand proofs and thus blame learners for being "stupid". This tendency is exacerbated by the contrast between learners who experience difficulty in understanding the origins of a theorem's proof and a few 'gifted' learners who appear to achieve understanding through memorization and rote learning for examination purposes (De Villiers 2008; Mudaly 2007). The reasons for learners' poor performance in constructing proofs include presenting proofs directly from the textbook and teaching proofs to learners who are not at the appropriate van Hiele level (Mudaly 2007). The promotional requirements in South Africa in the middle school phase allow learners to progress to the next grade whilst having only attained a minimum pass mark of 30 percent in Mathematics. This means that learners are likely to move to the next grade without having mastered geometry tasks at the significant van Hiele levels.

The common feeling amongst authors is that the failure of learners to construct proofs in Euclidean Geometry lies in the means by which proofs are taught (De Villiers 2008; Mudaly 2007). A quick perusal of the different Mathematics textbooks reveal that proving is seen as a verification of the truth. For example, proving that the opposite sides of a parallelogram are equal is seen to be similar to that of verifying that they are equal. Some textbooks even phrase questions such as "verify that the opposite sides of a parallelogram are equal". In his study, De Villiers (1990) found that most Higher Education Diploma students believed that the function of a proof is to verify the truth of a statement. This method of teaching proofs has been criticized by many authors as it encourages rote learning to learners – a feature of the old, traditional method of teaching (Mudaly 2007).

De Villiers (2008) argued that teaching proofs using explanation can also be done using the "genetic" approach. This approach involves the utilization of proof heuristics in contrast to that of direct textbook methods. In addition, De Villiers (2008) argued that reasoning by analogy can help stimulate the desire for proof understanding in learners. Analogy occurs when two figures share similar characteristics, for example, where the square is analogous to the cube as both have all sides equal (although the former is two-dimensional and the latter three dimensional). In this case, a cube is defined by constructive definition as a set of squares put together to form faces of a three-dimensional shape. However, De Villiers (2008) admits that this method of proving may be difficult to adopt as learners at school level are not likely to have been taught analogy. Perhaps the time will come when the contents of Mathematics textbooks might be transformed into ideas argued by authors who value proof by explanation.

Euclidean Geometry learning should be based on an understanding of the basic skills of the proofs of theorems. Examination questions are based on unseen problems in which learners cannot memorize facts but are instead required to apply their understanding of geometry riders. Mthembu (2007) argued that success in Euclidean Geometry depends on the knowledge of basic concepts as it is upon this knowledge that proofs and solutions of riders rely. Based on the researcher's experience, learners find it difficult to solve non-numerical riders.

According to Skemp (1976), 'understanding' may be divided into instrumental and relational. Instrumental understanding refers to knowing the procedures and laws of solving a problem. Relational understanding refers to both knowing the procedures and the reasons for choosing them in solving a particular problem. In the main, OBE supports relational understanding. Relational understanding in Mathematics seems to be difficult to achieve as some Mathematics teachers teach for the purposes of attaining instrumental understanding on the part of learners (Skemp 1976). It is also noted that the external examination for the school leavers consists mainly of questions that encourage instrumental understanding. Learner-centred approaches to teaching could be integrated into assessment in order that teachers might be encouraged to assimilate them into their teaching.

Learning Congruent Triangles

Congruent triangles are introduced to learners in middle schools (grade seven) in South Africa. The knowledge of congruent triangles is important when learners are preparing for high school geometry riders. Mathematics teachers, many of whom may have mastered the current learner-centred approach to teaching, teach congruency in group settings using the "cut-and-match" method of teaching identical triangles. However, formal proofs of the four cases of congruency are mostly taught as ready-made theorems in the textbook. Luthuli (1996) asserts that geometry teaching should be more learner-centred and closely based on learners' experiences. Constructivists like Cobb (1994) and Von Glasersfeld (1984), also argued for learner-participation in Mathematics lessons, where learners reflect on their learning and are able to communicate their thoughts to one another and to the teacher.

METHODOLOGY

Written responses and interviews were the data-capture instruments used in this research. The eighty-two participants were first handed out questionnaires containing two structured worksheets. The first worksheet was for Euclidean Geometry and the second was for Transformation Geometry. After the researcher had rated and analyzed the responses to both questionnaires, ten participants from each school were each interviewed about their performance in the two tasks. These ten learners were chosen two from each of the following mark ranges: 40 percent – 49 percent, 50 percent – 59 percent, 60 percent – 69 percent, 70 percent -79 percent and 80 percent – 100percent which they had attained in their previous years mathematics examination. The philosophy impinging the design of the two questionnaires is now presented.

Questionnaire 1: Euclidean Geometry

This questionnaire (see appendix) was a Problem-Centred Learning (PCL) activity involving congruent triangles. The researcher decided to implement the PCL with its featuring socio-constructivist theory. Murray et al. (1992) argued that PCL engages learners actively in the process of acquiring knowledge. The learners also draw on past experiences and existing knowledge during learning. In this questionnaire, the learners used their past knowledge of basic geometry (like parallel lines, isosceles triangles and alternating angles on parallel lines). The questionnaire also involves a formal proof of congruent triangles. As Mudaly (2007) admits that it is difficult to relate formal proof to learners' past experiences, and the researcher decided to use a form of guided activity whereby learners filled in the correct answers in spaces left intentionally blank. In South African schools, learners are not guided when doing these proofs and the researcher suspects that this may result in greater difficulties in learning such proofs at this level of schooling. The researcher wanted to monitor how much relative success learners enjoyed in determining formal proofs of congruent triangles. Hence, this activity is based on proof with rigour. This kind of proving by verification has been criticised for its lack of improving learners' interest and understanding. The researcher then wanted to observe how learners would understand and be stimulated to work out these kinds of proofs. (De Villiers 2008; Mudaly 2007). De Villiers (2008) further suggested that it was about time teachers attempt to teach proofs by explanation using analogies and heuristics.

Each sub-section of this questionnaire explored the understanding of a specific case of congruency. In 1.1 the case of side, side, side (S, S, S) is involved. To prove this, learners needed to know that both pairs of opposite sides of a parallelogram are equal. For 1.2 the learners required understanding of an isosceles triangle and the case of side, angle, side (S, A, S). For 1.3 the case of 90^0, hypotenuse, side (90^0, H, S) is deduced. Lastly, in 1.4 the case of angle, angle, side (A, A, S) is tested and the learners needed to understand properties and results derived from parallel lines and alternate angles.

The questionnaire was structured so as to lead learners into deducing the cases of congruency. The items in this questionnaire rely on the knowledge of the properties of triangles, parallelograms, parallel lines and quadrilaterals for successful solving. The questionnaire seeks the understanding of proofs using formal reasoning.

Questionnaire 2: Transformation Geometry

This questionnaire (see appendix) was based on Transformation Geometry and the tasks were based on the informal proofs of Geometry riders. The learners were guided through a practical activity where they would measure sides and angles to prove congruency. Initially, the learners were asked to draw the figure and its image on the same Cartesian plane under some kind of transformation, *viz.* translation, rotation or reflection. If the drawn image did not result in some kind of constructive defining, which De Villiers (2008) describes as a new concept created from the original figure, then the two triangles would be congruent. It is important for this activity to successfully stimulate learners' interest as Ausabel et al. (1978) argued that meaningful learning occurs when learners are curious to determine the result in a discovery task. This activity was a guided activity as research has shown guided learning facilitates an increase in learners' discovery (Mudaly 2007).

The purpose of these tasks was to check how much relative success learners achieve in doing informal proofs of riders. It is based on the knowledge of the four cases of congruency of triangles. In (a) the case of S, S, S is tested. To successfully complete this task, the learners need to know the rules of translation of points on a Cartesian plane as well as practical measurements of lines using a ruler. In (b) the learners were tested on the case of S, A, S. The knowledge of the rules of reflection on the Cartesian plane, as well as the ability to measure sides and angles was required in order to answer this question. For (c) the case of A, A, S was tested. The learners need the knowledge of the rules of rotation on the Cartesian plane as well as measurement of sides and angles. Lastly, for (d) the case of 90^0, H, S was tested along with the learners' knowledge of the rules of reflection and measurements.

OBSERVATIONS AND DISCUSSION

Two questionnaires were scaffolded using the guidelines proposed by Zhao and Orey (as cited in Lipscombe et al. 2004). The first questionnaire dealt with four tasks, each based on one case of congruency of triangles. This questionnaire was on formal reasoning in Euclidean geometry. The second questionnaire dealt with informal reasoning and had four tasks employing techniques from Transformation Geometry. We shall look at a task from each activity sheet and show the overall analysis of results of the two schools.

Formal Reasoning Task

A convenient coding system was employed in rating the quality of responses in both schools. Tasks 1 to 4 were marked out of 8 marks each. In the formal reasoning tasks, a rating scale of 0 to 8 was categorized as indicated by Table 1.

Table 1: Coding mechanism

Mark range	*Category*	*Descriptor*
0-2	A	Poor
3-5	B	Average
6-8	C	Good

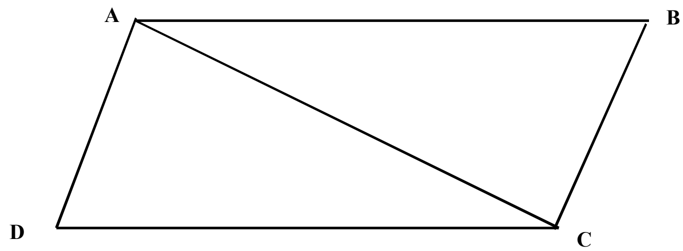
The results for the two schools in this questionnaire appear in Table 2. The data for those learners who were interviewed are shown.

Table 2: Performance of interviewed learners for tasks on formal reasoning

Mark range	*0-2*	*3-5*	*6-8*
School A - Task 1		10	
School B - Task 1		3	7
School A - Task 2	7	2	1
School B - Task 2		5	5
School A - Task 3	1	7	2
School B - Task 3		3	7
School A - Task 4	3	6	1
School B - Task 4		3	7

The data illustrate that the results for School B was better than those for School A in the 6 – 8 mark range. In these instances the better performance was due to the written responses showing correct reasons. For school A, all learners scored in Category B in task 1. On viewing the responses, it was observed that learners could provide the correct sides involved. They could all indicate AB = DC, AD = BC and AC = AC. However, the reasons provided were incorrect as shown by the response for learner A_3, presented in Figure 1.

Learner A_3 displayed the misconception that a pair of parallel lines in a quadrilateral imply equality of the length of the lines by using the symbols "=" and "//", as seen in Figure 1. This misconception could possibly be due to the trend that most applications of Euclidean Geometry provide quadrilateral figures (like rectangle, square rhombus and parallelogram), which involve opposite sides equal and parallel. This may lead learners to believe that when one condition is satisfied then the other is automatic. Further, as is unusual for learners to deal with application tasks involving trapeziums, these tasks may well have exposed the potential lack of comprehension.



Prove tht AADC=ACBA by completing the following statements.

Statement	Reason
AB = DC	(.................A.B.11 DC.............................)
AD = BC	(.................A.D.11 DC.............................)
AC=AC	(.................A.B.11 DC.............................)
ΔADC=ACBA	(.................5 5 5.............................)
	4/8

Fig. 1. Written response for learner A_3

Indeed, this specific quadrilateral is ideal to eradicate the above misconception, since it has exactly one pair of opposite sides parallel and not equal. In addition, this example could highlight the view that parallel sides need not be equal. To eradicate the converse we could promote the case of an isosceles triangle XYZ with XY = YZ by observing that XY cannot be parallel to YZ.

A positive note was that (for most of the learners), learners could identify the correct case of congruency. This could be that they were "scaffolded" into the correct conclusion by the structure of the task.

In order to triangulate the data, an interview was carried out with learner :

Researcher: *"What did you mean when giving this reason?"* (Showing the reason.)

Learner : *"AB is parallel to DC."*

Researcher: *"So that means that AB = DC?"*

Learner : *"Ja, like this."* (pointing to the given diagram.)

Researcher: *"So if the lines are parallel then they are also equal?"*

Learner: *"Ja like this. I don't know. Am I wrong? They look to be equal."*

(pointing to the diagram again.)

The last response raises another misconception that if the opposite sides of a quadrilateral (like a parallelogram, a rectangle, a square or a rhombus) look like they are equal, then it is true that they are equal without a formal reason. This behaviour seems common in the researcher's personal experience of teaching; especially in junior classes at high school level. Learners appear to prefer using visualisation instead of mental construction. This usage may be a factor which contributes to difficulty in proving Euclidean Geometry theorems and riders as concluded by Mudaly (2007).

Informal Reasoning Task

A convenient coding system was used to rate the quality of responses in both schools for the questionnaire addressing informal reasoning. Tasks 1 to 4 were marked out of 12 marks each. The performance of learners was rated from 0 to 12 as shown in Table 3.

Table 3: Coding mechanism

Mark range	*Category*	*Descriptor*
0-3	A	Poor
4-6	B	Average
7-9	C	Good
10-12	D	Excellent

The results for the two schools in this questionnaire follow in Table 4.

The general trend was that School A learners performed better in the 10 – 12 mark range.

In task one, fifty percent of the learners from school A scored in Category D, while forty percent scored in Category B. Most of the learners were able to do draw the diagram and its image on the Cartesian plane. A negative aspect of the responses was that some learners presented

Table 4: Performance of interviewed learners for tasks on informal reasoning

Mark range	*0-3*	*4-6*	*7-9*	*10 - 12*
School A-Task 1		4	1	5
School B-Task 1	4	4	2	
School A-Task 2		2	4	4
School B-Task 2	4	4	2	
School A-Task 3	2	5	1	2
School B-Task 3	7		3	
School A-Task 4	2	3	5	
School B-Task 4	7		3	

incorrect measurements of the sides and were thus unable to see that the corresponding sides of the two triangles were equal, contrary to the intention of the questionnaire.

Most learners correctly drew both diagrams on a Cartesian plane with the result that they may have been successful in learning reflections in their study of transformations. Learner seemed to be unable to use a protractor when measuring angles. This is another potential example of the importance of doing practical measurements of angles at lower standards. Learner could not provide a reason for concluding congruency. However, when the written responses of for questionnaire 1 were studied, we found that provided reasons for all tasks. This might have occurred since children might think that the requirements for congruency in Euclidean Geometry are not the same as congruency in Transformation Geometry. It is therefore important that all sections in mathematics teaching be linked (especially when a concept like congruency could be taught using more than one branch of mathematics).

The discussion between the researcher and learner is presented below (Fig. 2):

Researcher: *"Do you enjoy learning Euclidean Geometry? Why?"*

Learner : *"Yes, because we measure angles and side."*

Researcher: *"Do you feel that teachers can use other ways to teach Euclidean Geometry? Explain."*

Learner : *"Yes, by making more examples so that we understand."*

Researcher: *"In questionnaire number two, what types of transformations are displayed in each case?"*

Learner : *"Translation; reflection; rotation and reflection."*

Researcher: *"Which type of transformation is the easiest to work with?"*

Learner : *"Reflection not complicated."*

Researcher: *"Would you prefer using Transformation Geometry when ideas in Euclidean Geometry?"*

Learner *: "yes."*

From this dialogue, learner seemed to be enjoying Transformation Geometry and congruent triangles. This learner did not correctly measure the size of the angles but did indicate that he enjoyed performing such activities in Euclidean Geometry. Again, learner may have been treating Euclidean and Transformation Geometry as two disjoint branches in mathematics.

The analysis of the responses of the learners in questionnaire 2 appears to show that, depending on how well learners understand Transformation Geometry, learners may be capable of

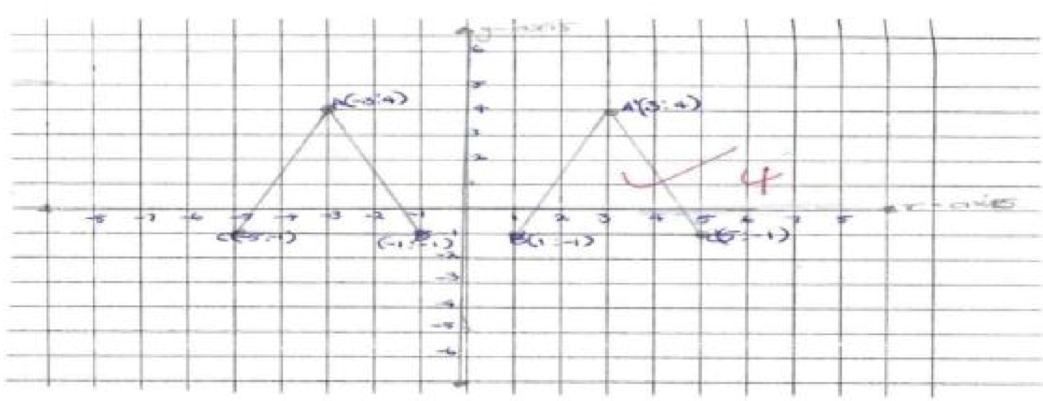


Fig. 2. Written response for learner

successfully learning congruent triangles. The researcher was interested in finding the comparative mean performance of learners in both schools for each task as this would possibly provide an overview of how successful learners had been in understanding Transformation Geometry through their learning of the congruency of triangles.

The overall performances in Euclidean Geometry, as well as in congruent triangles, could also be observed. Table 5 on the next page shows comparatively the mean performance of the two schools in each task of both formal and informal reasoning.

For each task, School B did well in questionnaire one (involving Euclidean Geometry) and did poorly in questionnaire two (involving Transformation Geometry). The converse appeared true for School A learners. The researcher believes that possibly there were some instructional differences of mathematics in the two schools in terms of Euclidean and Transformation Geometry teaching as shown in Table 6 where the overall mean performances of the two schools are comparatively analyzed for the two questionnaires:

Table 5: Mean performance of schools for each respective task on both questionnaires

Mean	*Formal reasoning*	*Informal reasoning*
School A - Task 1	4	8
School B - Task 1	7	4
School A - Task 2	2,3	6
School B - Task 2	6	4
School A - Task 3	4,3	9
School B - Task 3	7	5,2
School A - Task 4	8	8
School B - Task 4	6,2	4

The mean performance for school A in formal reasoning is almost two times lower than their mean performance in informal reasoning in all the tasks whilst the mean performance for school B in formal reasoning is almost two times higher that their mean performance in informal reasoning.

Most School A learners were able to correctly locate triangles on the Cartesian plane which likely means that the learners had an understanding of the points in the coordinate system. The learners could also locate the image of the triangle after a performed transformation, meaning that the learners had an understanding of translation, rotation and reflection. Teachers would need to strengthen learners' knowledge on Transformation Geometry as this could contribute to an improvement of their performance in the final year in high school. Some School A learners were unable to prove congruency in a convincing manner as they failed to correctly measure the sides and angles of the located triangles. Practical measurements of sides and angles are introduced at Senior Phase levels. Hence failure for learners to find correct measurements is an indication of the knowledge gaps present in learners upon their entry to grade ten. Improved facilitation and supervision of the performance of high school teachers in mathematics teaching could potentially therefore minimize this problem.

Table 6: Overall mean performance of the two schools for both questionnaires

School	*Mean performance: Formal reasoning*	*Mean performance: Informal reasoning*
School A	11.9	28.7
School B	23.8	13.3

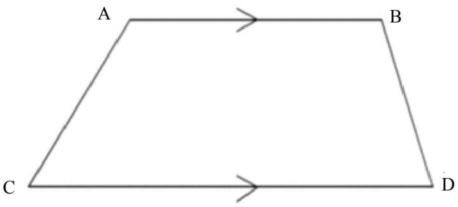
Most School B learners could correctly locate triangles after having performed transformation on the Cartesian plane. This appears to indicate that the learners were familiar with the coordinate system. However, most learners could not locate the image after having performed transformation on a Cartesian plane. This means that the comprehension of translation, rotation and reflection of the learners needs to be improved. This understanding may be improved by encouraging mathematics teachers to conduct their reflection after lessons in order for them to identify the strengths and weaknesses of learners. This could potentially aid in making an informed decision about the strategies of improvement for their Integrated Quality Management System (IQMS). The IQMS is a mechanism employed in the South African education system to ensure professional teacher growth.

School B learners who were successful in proving congruency used their knowledge of Analytical Geometry. It was encouraging to find that the learners were able to calculate distances as this would help them in their NSC tasks. However, knowledge gaps were evident when learners were unable to find the measurements of angles. These knowledge gaps may have contributed to lower overall mean learner performance as shown in Table 6. This creates opportunity for future research on exploring links in learning mathematics concepts in Analytical Geometry, Euclidean Geometry and Transformation Geometry.

In both schools confusion of the kinds of transformation in the tasks was observed, this was especially noticeable in the reflection along the x- axis and the reflection along the y-axis. It is hoped that this confusion could be resolved at the early stages of mathematical development. Mathematics teachers should adhere to teaching methods involving similar tasks as well as providing appropriate feedback after assessing these tasks. Of course providing relevant feedback depends on the pedagogical knowledge of these teachers and suggestions are made by Bansilal et al. (2014) and Brijlall and Maharaj (2015). Category D interviewed learners in both schools performed well in questionnaire 2. It is hoped that the introduction of Transformation Geometry in grade ten could then contribute to the improvement of the understanding of congruent triangles and vice versa.

CONCLUSION

In the course of this study, it was found that learners from school A confuse parallel and equal lines; the learners displayed the misconception that a pair of parallel lines in a quadrilateral imply equality of the length of the lines by using the symbols "=" and "//". According to discussions with teachers in the district cluster workshops, this appears to be common practice in mathematics classrooms. At the introductory phase of the teaching of quadrilaterals, it may be more effective to provide a larger number of examples with one condition, without the other. The examples in Figure 3 could be used:



In the above case learners could be asked: Is "AB=DC?

(ii)

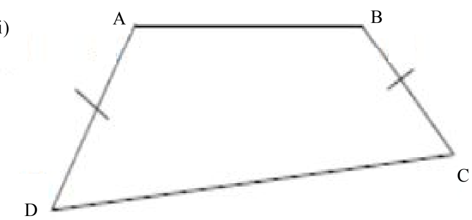


Fig. 3. Examples to enforce relationships between equality and parallel

In the above case where AD = BC learners could be asked to discuss whether AD is parallel to BC.

In school B, learners appeared to use improper geometrical reasoning. When they were given a quadrilateral as a parallelogram, the learners argued that opposite sides are equal because it was given, rather than using the reason: "opposite sides of parallelogram are equal". The learners' method of reasoning thus left the researcher doubting whether they properly understood the properties of a parallelogram. When engaging learners with tasks that involve the properties of a quadrilateral, teachers would need to first emphasize the procedure for solving riders. Learners appeared to be unable to use the reasoning of "given" properly. More examples of riders involving the reasoning of "given" and the properties of quadrilaterals may help clarifying this mistake.

It was found that learners prefer using visualization to arrive at conclusions rather than mental construction. This may be a factor contributing to the difficulty in proving Euclidean Geometry theorems and riders. This was observed particularly when a learner indicated that a corner of a house is equal to 90^0. If teachers were to teach more examples of angles that are very close to a 90^0 (like 92^0 or 89^0); it may well help resolve this misconception. Other potential examples could include quadrilaterals that look like a square, a rhombus, parallelogram or a rectangle. Learners could be asked to measure the lengths of the opposite sides of these figures so that they could see that these figures are not as they appear to be.

In number three of the questionnaire, very few learners who participated in the research could correctly answer and identify the case of 90^0, H, S in proving congruency. This would seem to indicate that learners are not as familiar with this case of congruency as they should be at grade ten. A quick perusal of mathematics textbooks reveals that even mathematics authors do not appear to include as many examples of the 90^0,H, S as other cases like S, S, S; S, A, S and A, A, S. Experience has also shown that even in the classroom this case of congruency is rarely done; probably because most teachers rely on textbooks when preparing for lessons.

It was also found that assessing learners' knowledge of congruent triangles in the context of quadrilaterals (like a parallelogram, a rhombus or a rectangle) does not give a clear indication of the learners' knowledge of congruent triangles. It was observed in some cases that learners did not possess adequate knowledge of the properties of quadrilaterals and hence could not prove congruency. This would seem to be an indication of the knowledge gaps learners develop when leaving for the next grade. This appears likely to result in high levels of rote learning; especially in concepts like the four cases of congruency. More time needs to be spent on teaching of the basic properties of figures like regular quadrilaterals and triangles at the appropriate level of learning.

The overall finding in this research is that school B performed almost two times better than school A in the formal tasks. This was deduced from the mean performance in each task, as well as the overall mean performance on the whole questionnaire illustrated in Figure 6. An inappropriate van Hiele level could be the cause of poor performance of learners in this questionnaire. This was apparent as learners could not answer questions related to alternate angles while proving congruency. One can then conclude that informal reasoning could be introduced to the learners while they are studying Euclidean Geometry especially the properties of geometrical shapes and patterns. While this is done the teachers has to ensure that learners are at the appropriate van Hiele level for that particular grade.

Recalling the critical research question, we acknowledge that the children's written responses and the analysis thereof do not convincingly identify one type of reasoning advantageous over the other. Also, this is a case study and so results cannot be generalized. However, those pedagogical implications which arose from the data were rich and make reliable recommendations for the teaching and learning of congruency of triangles. Also, it was found that the intervention of the questionnaires effectively managed to integrate the two types of reasoning.

RECOMMENDATIONS

The researcher recommends that the editors of mathematics textbooks rectify this situation by including as many of these types of cases as possible. Many mathematics teachers (especially inexperienced teachers) are likely to rely inordinately upon the use of textbooks in their teaching. With this recommendation, the researcher hopes other misconceptions around this case could be solved. For example, other learners involved in this study confused the 90^0,H, S case with the S, A, S case. Teachers need to emphasize that the angle in this case is the included one. Thorough teaching of congruency using relevant examples could overcome this problem. For example the learners can be asked to attempt the task in Figure 4.

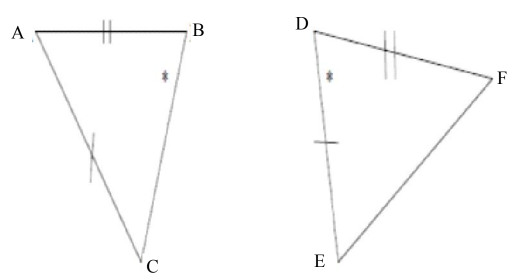


Fig. 4. Discussing whether the two triangles are always congruent

Some learners in task four seemed to believe that all alternate angles are equal in a quadrilateral. This was observed when learners indicated a pair of alternate angles equal when this was not the case. The researcher is of the view that this problem is caused by the tendency to introduce alternate angles (only) when teaching parallel lines, (and hence in this case the angles are equal). Experience has shown that very few teachers expose learners to alternate angles that are not equal. This problem could be a factor contributing to the poor performance of learners in National Senior Certificate assessment (final high school examination), and one is led to suspect that the same may well happen for corresponding and co-interior angles. Teachers may need to provide more counter-examples before introducing the concept of parallel lines. An example of this appears in Figure 5. The learners could be asked to name a pair of alternate angles in Figure 5, measure each of these alternate angles and discuss their relationship.

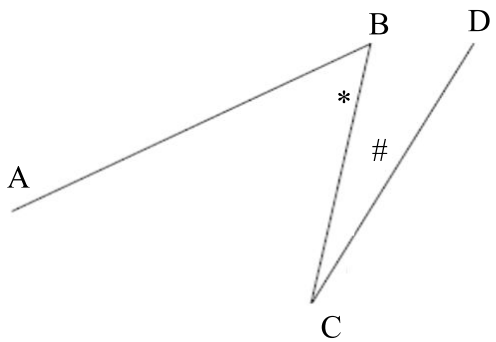


Fig. 5. Illustrating that not all alternate angles are equal

ACKNOWLDGEMENTS

I place on record my sincere appreciation for the proofreading of this paper by my colleague Mrs E Dowlath. I also acknowledge the data collection done by Mr Aaron Mbili.

REFERENCES

Adler J 2005. Mathematics for teaching: What it is and why it is important that we talk about it? *Pythagoras,* 62: 2-11.

Ausabel DP, Novak JD, Hanesian H 1978. *Educational Psychology: A cognitive View.* New York: Holt, Rinehart & Winston.

Bansilal S, Brijlall D, Mkhwanazi T 2014. An exploration of the common content knowledge of high mathematics Teachers. *Perspectives in Education,* 32(1): 34-50.

Beevers BS 2001. A Simpler Approach to Similar Triangles. Mathematics Teaching. From <http://www. ucpress.edu.edu/j.> (Retrieved on 23 February 2008).

Brijlall D 2011. A case study of lecturers' views on content knowledge and practice. In: T Mamiala, F Kwayasi (Eds.): *Proceedings of the Nineteenth Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education,* North-West University, 12-15 January, 2011, Mafekeng, South Africa, pp. 34-50.

Brijlall D, Bansilal S, Moore-Russo D 2012. Exploring teachers' conceptions of representations in mathematics through the lens of positive deliberative interaction. *Pythagoras*, 33(2): Art.#165, 8 pages. http://dx.doi.org/10,4102/pythagoras.v33i2.165

Brijlall D, Isaac V 2011. Links between content knowledge and practice in a mathematics teacher education course: A case study. *South African Journal of Higher Education*, 25(4): 680-699.

Brijlall D, Maharaj A 2011. Using an inductive approach for definition making: Monotonicity and boundedness of Sequences. *Pythagoras*, 70: 68 – 79.

Brijlall D, Maharaj A 2014. Exploring support strategies for high school mathematics teachers from underachieving schools. *International Journal of Educational Sciences*, 7(1): 99-107.

Brijlall D, Maharaj A 2015. Exploring pre-service teachers' mental constructions when solving problems involving infinite sets. *International Journal of Educational Sciences*, 9(3): 265-271.

Brijlall D, Maharaj A, Jojo ZMM 2006. The development of geometrical concepts through design activities during Technology education class. *African Journal Research in SMT Education,* 10(1): 237-245.

Cobb P 1994. Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher,* 23(7): 13-20.

Davis PJ, Hersh R 1983. *The Mathematical Experience.* Boston, New York: Houter Mifflin Co.

Department of Education 2003. *National Curriculum Statement Grades 7-9*. Pretoria: Department of Education.

De Villiers M 2008. Generalizing the Nagel line to circumscribed polygons by analogy and constructive defining. *Pythagoras,* 68: 32-40.

De Villiers M 1990. The role and function of proof in mathematics. *Pythagoras,* 24: 17- 24.

Even R 1990. Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics,* 21(6): 521-544.

Fagnant A 2005. The use of mathematical symbolism in problem solving: An empirical study carried out in grade one in the French community of Belgium. *European Journal of Psychology of Education,* 20(4): 355-365.

Fernandez S 2005. Transformation Geometry. *Mathematics Teaching,* 191: 27-29.

Hall RW 1999. *A Course in Multicultural Mathematics*. USA: Saint Joseph's University.

KwaZulu-Natal Department of Education 2004. *Senior Certificate Examiners' Report: Mathematics*. Durban: Department of Education.

Lipscombe L, Swanson J, West A 2004. Scaffolding: From Emerging Perspectives on Learning, Teaching and technology. From <http://projects.coe. uga. edu>. (Retrieved on 29 October 2008).

Luthuli DV 1996. Questions, reflections and problem posing as sources of inquiry in Euclidean geometry. *Pythagoras,* 40: 17-27.

Mthembu SG 2007. *Instructional Approaches in the Teaching of Euclidean Geometry in Grade 11*. Master Dissertation, Unpublished. In partial fulfillment of the requirements for the Degree of Master of Education (Mathematics Education). Durban: University of KwaZulu-Natal.

Mudaly V 2007. Proof and proving in secondary school. *Pythagoras,* 6: 64-74.

Murray H, Oliver A, Human P 1992. *The Development of Young Students' Division Strategies*. South Africa: University of Stellenbosch.

Ndlovu Z, Brijlall D 2016. Preservice Mathematics Teachers' mental constructions of the determinant concept. *International Journal of Educational Sciences*, 14(12): 145-156.

Rival I 1987. Picture Puzzling: Mathematics are rediscovering the power of pictorial reasoning. *The Sciences,* 27: 41-46.

Skemp RR 1976. Relational understanding and instrumental understanding. *Mathematics Teaching,* 77: 20-26.

Van Hiele PM 1999. Developing geometric thinking through activities that begin with play. *Teaching Children Mathematics,* 5(6): 310-317.

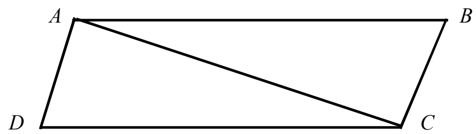
Von Glasersfeld E1984. An introduction to radical constructivism. In P. Watzlawick (Ed.): *The Invented Reality.* Norton: New York, pp. 75-92.

Paper received for publication on October 2014
Paper accepted for publication on November 2016

APPENDIX

Questionnaire 1: Euclidean Geometry

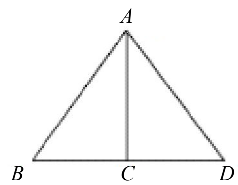
1.1 Below is parallelogram ABCD.



Prove that ΔADC ≡ Δ CBA by completing the following statements:

Statement Reason
AB = ... (..)
AD = ... (..)
AC = ... (..)
So, Δ ADC ≡ Δ CBA (... ....)

1.2 Below is isosceles Δ ABC with AB = AC. DA bisects Â.



Prove that Δ ABD ≡ ΔACD by completing the following statements:

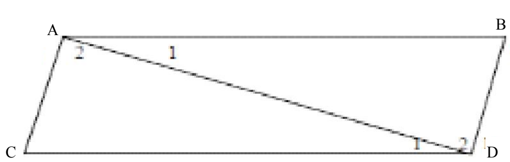
Statement Reason
..... = ... (given)
$\hat{A}_1$ =.... (..)
AD=AD (..)
So, Δ ABD≡Δ ADC (...)

1.3 Below is quadrilateral ABCD with Â = Ĉ = 90^0 and AB = DC.

Prove that ΔABD ≡ ΔCDB by completing the following statements:

Statement Reason
Â = ... (..)
BD = BD (..)
AB = ... (..)
So, ΔABD ≡ Δ CDB (..)

1.4 Below is quadrilateral ABCD with Â = C and AB // CD.

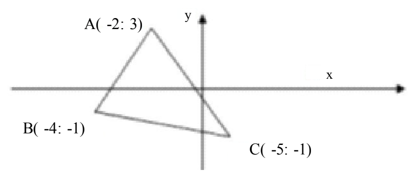


Prove that ΔADB ≡ ΔCBD by completing the following statements:

Statement Reason
Â = ... (..)
... = ... (Alternate angles, AB // DC)
BD = BD (..)
So, ΔADB ≡ΔCBD (..)

Questionnaire 2: Transformation Geometry

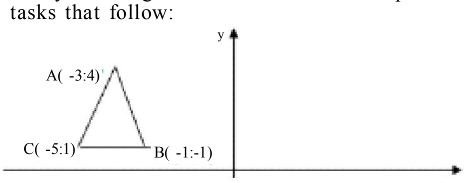
Study the diagram below and then complete the tasks that follow:



Δ ABC is translated by the rule: (x, y) → (x + 1, y + 2). Draw both ΔABC and its image ΔA'B'C' on the same set of axis on the graph paper provided. Complete the following statements by accurately measuring:

AB = ... A' B' = ...
BC = ...B' C' = ...
AC = ...A'C' = ...
So ΔABC ≡ ΔA'B'C' (.............................)

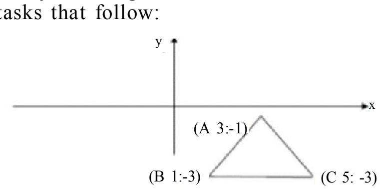
(b) Study the diagram below and then complete the tasks that follow:



In the above diagram, ΔABC is reflected about the y-axis. Draw both of ΔABC and its image ΔA´B´C´ on the same set of axis on the graph paper provided. Complete the following statements:

AB = ...A'B' = ...
Â = ...Â' = ...
BC = ...B'C' = ...
So ΔABC ≡ ΔA' B'C' (.............................)

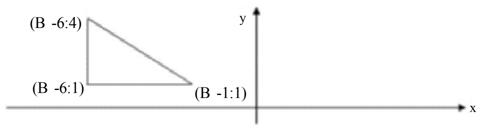
(c) Study the diagram below and then complete the tasks that follow:



In the above diagram, ΔABC has been rotated through 180^0 clockwise. Draw both ΔABC and its image ΔA'B'C' on the same set of axis on the graph paper provided.Complete the following statements by accurately measuring:

Â = ……................ Â2 = ...
C = ……................ C' = ...….........
BC = ……................ B'C' = ...….........
So, ΔABC ≡ ΔA'B'C' (......... ....……………..)

(a) Study the diagram and then complete the tasks that follow:



In the figure above, ÄABC has been reflected along the x-axis. Draw both ΔABC and its image ΔA´B´C´ on the same set of axis.Complete the following statements by accurately measuring:
AC = ……....... A'C' = ...…
B^2 = ...…....B' = ...….……….
BC =...............B' C' = ...…
So ΔABC ≡ ΔA'B'C' (.... …….………..)

Semi-structured Interview Questions

(a) Do you enjoy learning Euclidean Geometry? Why?
(b) Do you feel that the teachers can use other ways to teach Euclidean Geometry? Explain.
(c) Do you think that most learners enjoy working on problems involving congruency? Why do you think this is so?
(d) Tell me what we mean when we say "the two triangles are congruent"?
(e) Can you explain the cases of congruency? How many cases are there?
(f) Give me an example of two triangles that are not congruent.
(g) Can you tell me in Questionnaire number 2, what types of Transformation are displayed in each case?
(h) In which tasks did you enjoy working with? Why?
(i) Would you prefer using Transformation Geometry when studying ideas in Euclidean Geometry?